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Green's function of the electromagnetic field in biaxial media

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Asymptotic properties of the Green's function of an electromagnetic field in the far zone of biaxial anisotropic media are examined, based on ideas proposed by Lax and Nelson [Phys. Rev. B **4**, 3694 (1971)]. The rather complicated structure of the wave surface and the ray surface, in particular the existence of their singular points, is taken into account. Starting from a detailed analysis of the wave-surface Gaussian curvature, we find the directions of Green's-function asymptotic behavior differing from the usual R^{-1} relationship. These directions are the directions along the biradials of a biaxial medium, and the infinite sets of directions defined by a wave vector directed along every one of the binormals. In the first case, the asymptotical form of the Green's function is proportional to $R^{-1/2}$; in the second case, this asymptotical form is proportional to $R^{-5/4}$. A smooth transition from the asymptotic form proportional to $R^{-1/2}$ to the usual asymptotic form is analyzed. The possibility of an experimental observation of this unusual asymptotic behavior is discussed.

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I. INTRODUCTION

Modern methods of solving many optical problems are based on an integral formulation of the electromagnetic-field equations. Among these problems are scattering in random and determined inhomogeneities, diffraction problems, wave propagation in wave guides, etc. The integral formulation requires a determination of the electromagnetic-field Green's function—the field of a point source. It is well known for unlimited isotropic media. In this case, the Green's function at a large distance R is a diverging spherical wave whose amplitude decreases as R^{-1}. Difficulties appear if we begin to examine limited media or give up the condition of isotropy [1].

We are interested in the special feature of the Green's function in unlimited undispersing anisotropic media. In this case, the solution of the problem is rather complicated. A general approach was developed by Lax and Nelson [2,3]. In the case of uniaxial anisotropic media they reported an explicit expression for the Green's function [3]. It consists of two terms connected with ordinary and extraordinary waves whose amplitudes decrease as R^{-1} at a large distance. The first term is a spherical wave. An amplitude of the second term depends on the direction $\mathbf{r}=\mathbf{R}/R$, and its phase surface is an ellipsoid of revolution. The Green's function obtained for uniaxial media has permitted the solution of the light-scattering problem [4] and the calculation of the extinction coefficient [5,6] in an ordered phase of nematic liquid crystals. Interesting physical results has been obtained in this manner. There is, for example, a principal difference between ordinary- and extraordinary-beam propagation and scattering [7].

The case of biaxial media has not been studied in detail. However, it is necessary to investigate defects in biaxial crystals, fluctuations in some liquid crystals (biaxial nematic and smectic-C liquid crystals), etc.

In this paper we examine asymptotic properties of the Green's function in biaxial anisotropic media. In Sec. II well-known general facts from the theory of wave propagation in anisotropic media are presented. All relationships are given in the invariant form in terms of wave vectors and the ray vectors. The properties of the wave surface and the ray surface and their connection are considered. In Sec. III the expression for the Green's-function $(\omega,\mathbf{R})$ representation is obtained by the method developed by Lax and Nelson [3]. Some inaccuracies of their exposition are removed. A detailed analysis of the contribution to the Green's function from stationary points dependent on their situation on the wave surface is carried out. The existence of directions $\mathbf{R}$, for which the approach of Lax and Nelson is inapplicable, is shown (for these directions, the corresponding contributions to the Green's function turn into zero or into infinity).

In Secs. IV and V a method for analyzing the Green's-function $\hat{G}(\omega,\mathbf{R})$ asymptotic behavior in the vicinity of these special directions $\mathbf{R}$ is suggested. It is shown that these asymptotic forms are not proportional to R^{-1} (they are proportional to $R^{-1/2}$ or $R^{-5/4}$). The existence of these unusual asymptotic behaviors for the field of the point source is analogous to external and internal conic refraction effects well known for plane waves in biaxial media.

II. OPTICAL PROPERTIES OF BIAXIAL MEDIA

From the system of Maxwell's equations for homogeneous nonmagnetic media we can obtain

$$[k^2(\hat{I}-\mathbf{s}\otimes\mathbf{s})-(\omega/c)^2\hat{\varepsilon}]\mathbf{E}(\omega,\mathbf{k})=4\pi(\omega/c)^2\mathbf{P}(\omega,\mathbf{k}) \ , \quad (2.1)$$

where $\mathbf{k}$ is the wave vector, $\mathbf{s}=\mathbf{k}/k$, ω is the circular frequency, $\mathbf{E}$ is the electric field intensity, $\mathbf{P}$ is the vector of polarization induced by sources, $\hat{\varepsilon}$ is the dielectric tensor, and $\hat{I}$ is the unit tensor. Here $\mathbf{a}\otimes\mathbf{b}$ is the tensor product of the vectors $\mathbf{a}$, $\mathbf{b}$, i.e., the tensor with components $a_\alpha b_\beta$. The corresponding homogeneous system

$$[n^2(\hat{I}-\mathbf{s}\otimes\mathbf{s})-\hat{\varepsilon}]\mathbf{E}(\omega,\mathbf{k})=0\ , \tag{2.2}$$

where $n=|\mathbf{n}|$ and the vector $\mathbf{n}$ is determined by the relationship $\mathbf{k}=(\omega/c)\mathbf{n}$, describes the normal waves that can propagate in the media with dielectric tensor $\hat{\varepsilon}$. The dispersion equation (Fresnel's equation) expressing the determinant of the system (2.2) equaling zero has the form

$$\det\hat{\varepsilon}^{-1}(\mathbf{s}\hat{\varepsilon}\mathbf{s})n^4-[\mathrm{Tr}\hat{\varepsilon}^{-1}-(\mathbf{s}\hat{\varepsilon}^{-1}\mathbf{s})]n^2+1=0\ . \tag{2.3}$$

Here we use the notation $(\mathbf{f}\hat{\tau}\mathbf{g})$ for the total convolution of vectors $\mathbf{f}$ and $\mathbf{g}$ and tensor $\hat{\tau}$.

The Fresnel's equation has two roots in the general case,

$$n_{(j)}(\mathbf{s})=\left[\frac{\mathrm{Tr}\hat{\varepsilon}^{-1}-(\mathbf{s}\hat{\varepsilon}^{-1}\mathbf{s})-(-1)^j\Delta_n^{1/2}(\mathbf{s})}{2\det\hat{\varepsilon}^{-1}(\mathbf{s}\hat{\varepsilon}\mathbf{s})}\right]^{1/2}\ , \tag{2.4}$$

where $j=1,2$ and

$$\Delta_n(\mathbf{s})=[\mathrm{Tr}\hat{\varepsilon}^{-1}-(\mathbf{s}\hat{\varepsilon}^{-1}\mathbf{s})]^2-4\det\hat{\varepsilon}^{-1}(\mathbf{s}\hat{\varepsilon}\mathbf{s})\ . \tag{2.5}$$

We see now that in the general case two normal waves with a common direction of the wave vector $\mathbf{s}$ can propagate in anisotropic media. Here and further the subscripts 1 and 2 label the values connected with these two waves. The absolute values of their wave vectors are defined by Eq. (2.4). The directions of their polarization vectors $\mathbf{e}_{(j)}=\mathbf{E}_{(j)}/E_{(j)}$ for these waves are given by the system (2.2). For biaxial media, we obtain

$$\mathbf{e}_{(j)}(\mathbf{s})\|(n_{(j)}(\mathbf{s})\hat{I}-\hat{\varepsilon})^{-1}\mathbf{s}\ . \tag{2.6}$$

Taking into account the theorem of Kelly and Hamilton from a matrix theory [8], we can rewrite Eq. (2.6) in a more convenient form,

$$\mathbf{e}_{(j)}(\mathbf{s})\|\hat{Q}_{(j)}(\mathbf{s})\mathbf{s}\ , \tag{2.7}$$

where the tensor $\hat{Q}_{(j)}(\mathbf{s})$ is defined by

$$\hat{Q}_{(j)}(\mathbf{s})=n_{(j)}^2\hat{\varepsilon}+n_{(j)}^2(n_{(j)}^2-\mathrm{Tr}\hat{\varepsilon})\hat{I}+(\det\hat{\varepsilon})\hat{\varepsilon}^{-1}\ . \tag{2.8}$$

Fresnel's equation (2.3) describes the wave surface. In the case of biaxial media, where the eigenvalues ε_1, ε_2, and ε_3 of the tensor $\hat{\varepsilon}$ are different, this surface self-intersects and has four singular points (the points of self-intersection). The directions $\mathbf{s}$ corresponding to these points define two axes called the binormals. It is convenient to imagine this surface as the union of two closed parts: the external part and the internal part. The subscript (1) corresponds to the external part and the subscript (2) corresponds to the internal part.

In analogy to the vector $\mathbf{n}$, a wave can be described in the terms of the ray vector $\mathbf{m}$. The direction of this vector coincides with that of Poynting's vector of the wave and, generally speaking, does not coincide with the direction of the vector $\mathbf{n}$. We define the absolute value m of the vector $\mathbf{m}$ as follows [9]:

$$(\mathbf{m}\cdot\mathbf{n})=1\ . \tag{2.9}$$

The angle δ between the vectors $\mathbf{n}$ and $\mathbf{m}$ for the wave propagating in an anisotropic medium is equal to the angle between the vector of the electric field intensity $\mathbf{E}$ and the vector of the electric field induction $\mathbf{D}=\hat{\varepsilon}\mathbf{E}$. This angle is given by

$$\cos\delta=\frac{1}{nm}=\frac{(\mathbf{e}\hat{\varepsilon}\mathbf{e})}{(\mathbf{e}\hat{\varepsilon}^2\mathbf{e})^{1/2}}\ . \tag{2.10}$$

There is some correspondence between ray vectors and wave vectors: the change

$$\hat{\varepsilon}\to\hat{\varepsilon}^{-1}\ ,\quad \mathbf{n}\to\mathbf{m}\ ,\quad \mathbf{E}\to\mathbf{D} \tag{2.11}$$

in any equation for the quantities $\hat{\varepsilon}$, $\mathbf{n}$, and $\mathbf{E}$ leads to an analogous correct equation [9]. In this way, it is easy to obtain the equation

$$\det\hat{\varepsilon}(\mathbf{r}\hat{\varepsilon}^{-1}\mathbf{r})m^4-[\mathrm{Tr}\hat{\varepsilon}-(\mathbf{r}\hat{\varepsilon}\mathbf{r})]m^2+1=0\ , \tag{2.12}$$

which is analogous to Eq. (2.3). Here the unit vector $\mathbf{r}=\mathbf{m}/m$ has been introduced. Equation (2.12) has two roots in the general case,

$$m^{(j)}(\mathbf{r})=\left[\frac{\mathrm{Tr}\hat{\varepsilon}-(\mathbf{r}\hat{\varepsilon}\mathbf{r})+(-1)^j\Delta_m^{1/2}(\mathbf{r})}{2\det\hat{\varepsilon}(\mathbf{r}\hat{\varepsilon}^{-1}\mathbf{r})}\right]^{1/2}\ , \tag{2.13}$$

where $j=1,2$ and

$$\Delta_m(\mathbf{r})=[\mathrm{Tr}\hat{\varepsilon}-(\mathbf{r}\hat{\varepsilon}\mathbf{r})]^2-4\det\hat{\varepsilon}(\mathbf{r}\hat{\varepsilon}^{-1}\mathbf{r})\ . \tag{2.14}$$

Thus, there are two normal waves [$j=1$ and $j=2$ in Eq. (2.13)] for the defined direction $\mathbf{r}$ of a ray vector. Here and further we use the superscripts 1 and 2 for the designation of values corresponding to these two waves. The directions of the polarization vectors for them in a biaxial medium are given by

$$\mathbf{e}^{(j)}(\mathbf{r})\|\hat{Q}^{(j)}(\mathbf{r})\mathbf{r}\ , \tag{2.15}$$

where

$$Q^{(j)}(\mathbf{r})=(m^{(j)})^{-2}\hat{\varepsilon}+(m^{(j)})^{-2}[(m^{(j)})^{-2}-\mathrm{Tr}\hat{\varepsilon}]\hat{I}+(\det\hat{\varepsilon})\hat{\varepsilon}^{-1}\ . \tag{2.16}$$

Equation (2.12) describes the ray surface. The structure of the ray surface is analogous to that of the wave surface. Directions $\mathbf{r}$ corresponding to the common points of the internal and external ray-surface parts (the points of self-intersection) define two axes called the biradials. The superscript 2 in Eq. (2.13) is related to the external part of the ray surface and the superscript 1 is related to the internal part.

Every plane wave in a medium is characterized by its own vectors $\mathbf{n}$ and $\mathbf{m}$; moreover, in general, different wave or different ray vectors correspond to different plane waves. Thereby, there is some correspondence between wave-surface points and ray-surface points. If a wave is characterized by the ray vector $\mathbf{m}$, the wave-vector direction $\mathbf{s}=\mathbf{n}/n$ is defined as normal to the ray surface in the point determined by the vector $\mathbf{m}$. On the contrary, for a wave characterized by the vector $\mathbf{n}$, its propagation direction (i.e., the direction $\mathbf{r}=\mathbf{m}/m$ of its ray vector) is defined as normal to the wave surface in the point determined by the vector $\mathbf{n}$. The absolute values of these vectors (n in the first case and m in the second case) are defined by the relationship (2.9). The correspondence between points of the wave surface and those of the ray surface is one-to-one everywhere, except the singular points of each surface.

The connection between the Gaussian curvatures of the wave surface, K_{Gn}, and the ray surface, K_{Gm}, in the corresponding points is given by the relationship

$$K_{Gn}K_{Gm}=\cos^4\delta \tag{2.17}$$

(see the Appendix). Each surface Gaussian curvature turns into infinity at its singular points. Thus Eq. (2.17) shows that the Gaussian curvature turns into zero at the points of each surface corresponding to the singular points on the other surface. This can be easily seen from the fact that, for the media we are interested in, $\delta<\pi/2$. Each of the four singular points on any surface corresponds to the whole line of points (the circle) on the other surface. The affirmation about the existence of two types of waves, when the vector $\mathbf{s}$ is determined, is violated for the vectors $\mathbf{s}$ directed along a binormal. In this case, the whole cone of ray vectors, called the internal conic refraction cone, corresponds to this vector $\mathbf{s}$. Analogously, the affirmation about the existence of two types of waves, when the vector $\mathbf{r}$ is determined, is violated for the vector $\mathbf{r}$ directed along a biradial. In this case, the whole cone of wave vectors, called the external conic refraction cone, corresponds to this vector $\mathbf{r}$. The intersection of the external conic refraction cone with the wave surface and that of the internal conic refraction cone with the ray surface give the circles of the points of the Gaussian curvature equal to zero, described above.

The connection between the vectors $\mathbf{m}$ and $\mathbf{n}$ can be written in an analytic form. Taking into account the fact that the vector $\mathbf{n}$ is directed along the normal to the ray surface, we can find the vectors $\mathbf{n}^{(j)}(\mathbf{r})$ ($j=1,2$) corresponding to two normal waves with defined direction of the ray vector $\mathbf{r}$. By using Eqs. (2.9) and (2.12)–(2.14), we obtain

$$\mathbf{n}^{(j)}(\mathbf{r})=\hat{W}^{(j)}(\mathbf{r})\mathbf{r}\ , \tag{2.18}$$

where the tensor $\hat{W}^{(j)}(\mathbf{r})$ is given by

$$\hat{W}^{(j)}(\mathbf{r})=(-1)^j\frac{m^{(j)}}{\Delta_m^{1/2}}\hat{Q}^{(j)}(\mathbf{r})+\frac{1}{m^{(j)}}\hat{I}\ , \tag{2.19}$$

and $m^{(j)}$ and Δ_m are defined by Eqs. (2.13) and (2.14). Note that the vector $\mathbf{n}^{(j)}(\mathbf{r})$ is always situated at the plane of the vectors $\mathbf{e}^{(j)}(\mathbf{r})$ and $\mathbf{r}$ [9]. The two terms in Eqs. (2.19) and (2.18) are connected with the expansion of $\mathbf{n}^{(j)}$ by $\mathbf{r}=\hat{I}\mathbf{r}$ and $\mathbf{e}^{(j)}\|\hat{Q}^{(j)}\mathbf{r}$.

III. GREEN'S FUNCTION IN THE FAR ZONE: GENERAL CASE

We define the Green's function $\hat{G}(\omega,\mathbf{k})$ of an electromagnetic field by the equation

$$\mathbf{E}(\omega,\mathbf{k})=4\pi\hat{G}(\omega,\mathbf{k})\mathbf{P}(\omega,\mathbf{k})\ . \tag{3.1}$$

It was shown by Lax and Nelson [2] that it was possible to write the Green's function $\hat{G}(\omega,\mathbf{k})$ for an anisotropic medium in the form

$$\hat{G}(\omega,\mathbf{k})=(\omega/c)^2\sum_{j=1,2}\frac{\hat{\mathscr{E}}_{(j)}(\mathbf{s})}{\left[\dfrac{k}{n_{(j)}(\mathbf{s})}\right]^2-(\omega/c)^2-i0}-\frac{\mathbf{s}\otimes\mathbf{s}}{(\mathbf{s}\hat{\varepsilon}\mathbf{s})}\ . \tag{3.2}$$

Here

$$\hat{\mathscr{E}}_{(j)}(\mathbf{s})=\frac{e^{(j)}(\mathbf{s})\otimes\mathbf{e}_{(j)}(\mathbf{s})}{[\mathbf{e}_{(j)}(\mathbf{s})\hat{\varepsilon}\mathbf{e}_{(j)}(\mathbf{s})]}\ , \tag{3.3}$$

$\mathbf{n}_{(j)}(\mathbf{s})$ is defined by Eq. (2.4), and $\mathbf{e}_{(j)}(\mathbf{s})$ is defined by Eq. (2.7). The third item in Eq. (3.2) corresponds to the longitudinal wave. The term $-i0$ in the denominator of Eq. (3.2) is connected with causality. Here we consider $\omega>0$. If $\omega<0$, it is necessary to change the term $-i0$ to $+i0$.

If we are interested in the $(\omega,\mathbf{R})$ representation of the Green's function, we must make a Fourier transformation of Eq. (3.2),

$$\hat{G}(\omega,\mathbf{R})=\int\frac{d\mathbf{k}}{(2\pi)^3}\hat{G}(\omega,\mathbf{k})\exp\{i\mathbf{k}\cdot\mathbf{R}\}\ . \tag{3.4}$$

For many optical problems, the inequality $R\gg\lambda$ is valid. That is why we are interested below in an asymptotic form of $\hat{G}(\omega,\mathbf{R})$ at a large distance R. The asymptotic behavior of the main contribution in Eq. (3.4) is proportional to R^{-1} when $R\to\infty$. At the same time, the third term in Eq. (3.2) decreases as R^{-3} in the asymptotic expansion of $\hat{G}(\omega,\mathbf{R})$. (It corresponds to the static dipole field.) That is why we are interested in only the first and second terms of Eq. (3.2).

A residue integration of integral (3.4) over the component of $\mathbf{k}$ which is parallel to $\mathbf{R}$ and the transition in the resulting two-dimensional integral from transverse to $\mathbf{R}$ components of $\mathbf{k}$ to an integration over the surfaces $A_{(j)}$ [i.e., $k=k_{(j)}(\mathbf{s})$], $j=1,2$, yield

$$\hat{G}(\omega,\mathbf{R})=(\omega/c)^2\frac{\exp(i\pi/2)}{(2\pi)^2}\sum_{j=1,2}\int_{A_{(j)}^+}dA_{(j)}^+\frac{\hat{\mathscr{E}}_{(j)}(\mathbf{s})\exp\{i\mathbf{k}\cdot\mathbf{R}\}}{\left|\nabla_{\mathbf{k}}\left[\dfrac{k}{n_{(j)}(\mathbf{s})}\right]^2\right|}\ . \tag{3.5}$$

The terms $-i0$ in the denominators of Eq. (3.2) defining the circumvention rule lead to the integration in Eq. (3.5) not over the entire surface $A_{(j)}$ but over the part whose points satisfy the following condition: the projection of $\mathbf{k}_{(j)}$ at $\mathbf{R}$ is a positive value. We have designated this part as $A^{+}_{(j)}$. This requirement provides for the existence of diverging waves only in Eq. (3.5).

It was shown in [3] that

$$\left|\nabla_{\mathbf{k}}\left[\frac{k}{n_{(j)}(\mathbf{s})}\right]^2\right|=\frac{2k}{n^2_{(j)}(\mathbf{s})\cos\delta_{(j)}(\mathbf{s})} . \tag{3.6}$$

Thus,

$$\hat{G}(\omega,\mathbf{R})=(\omega/c)\frac{\exp\{i\pi/2\}}{8\pi^2}\sum_{j=1,2}\int_{A^{+}_{(j)}(u,v)} dA^{+}_{(j)}(u,v)n_{(j)}(u,v)\cos\delta_{(j)}(u,v)\hat{\mathscr{E}}_{(j)}(u,v)\exp\{i\mathbf{k}_{(j)}(u,v)\cdot\mathbf{R}\} , \tag{3.7}$$

where u and v are the parametrized variables of the surface of integration.

It is convenient to use a stationary-phase method to evaluate an integral (3.7) in the far zone. The stationary-phase condition

$$\frac{\partial\mathbf{k}_{(j)}}{\partial u}\cdot\mathbf{R}=\frac{\partial\mathbf{k}_{(j)}}{\partial v}\cdot\mathbf{R}=0 , \tag{3.8}$$

selects the stationary points $\mathbf{k}_{(j)}(\mathbf{r})$ on the surfaces of the integration, where the surface normals are parallel to the direction $\mathbf{r}=\mathbf{R}/R$. Actually, the surfaces of the integration in Eq. (3.7) are the external and internal parts of the wave surface. As it was shown in Sec. II, this surface normal direction coincides with the ray-vector direction, so the solutions of Eq. (3.8) are given by Eq. (2.18),

$$\mathbf{k}^{(j)}(\mathbf{r})=(\omega/c)\hat{W}^{(j)}(\mathbf{r})\mathbf{r} . \tag{3.9}$$

According to the stationary-phase method, it is necessary to calculate the asymptotic behavior of $\hat{G}(\omega,\mathbf{R})$ to expand the exponent in series up to the second-order approximation in the vicinity of the stationary points and to change the other terms to their corresponding values at the stationary points. The resulting Gaussian integral is easily calculated and the Green's function can finally be written (cf. [3]) as

$$\hat{G}(\omega,\mathbf{R})=\sum_{j=1,2}\hat{G}^{(j)}(\omega,\mathbf{R}) , \tag{3.10}$$

where

$$\hat{G}^{(j)}(\omega,\mathbf{R})=(\omega/c)^2\frac{\sigma^{(j)}_n}{4\pi R}\frac{n^{(j)}\cos\delta^{(j)}}{|K^{(j)}_{Gn}|^{1/2}}\hat{\mathscr{E}}^{(j)}\exp\{i\mathbf{k}^{(j)}\cdot\mathbf{R}\} , \tag{3.11}$$

$K^{(j)}_{Gn}$ are the wave-surface Gaussian curvatures at the stationary points $\mathbf{n}^{(j)}$, and the values $\sigma^{(j)}_n$ depend on the signs of the two principal curvatures λ_1 and λ_2 at the stationary points $\mathbf{n}^{(j)}$ on the wave surface: $\sigma=-1$ if $\lambda_1>0$, $\lambda_2>0$, $\sigma=1$ if $\lambda_1<0$, $\lambda_2<0$, and $\sigma=i$ if $\lambda_1\lambda_2<0$ (see [10], Appendix II). Here the value λ_i is considered to be positive if a shift from the stationary point along the principal direction corresponding to λ_i gets the value $\mathbf{k}^{(j)}\cdot\mathbf{r}$ increasing. In our case, the wave surface can only have two types of structure: $\lambda_1<0$, $\lambda_2<0$ (i.e., $\sigma=1$) and $\lambda_1\lambda_2<0$ (i.e., $\sigma=i$). Taking into account that $K_G=\lambda_1\lambda_2$ we can conclude that actually these two situations differ only in the signs of the Gaussian curvatures. Thus we can write for $\sigma^{(j)}_n$ in Eq. (3.11)

$$\sigma^{(j)}_n=\exp\{(i/2)\arg K^{(j)}_{Gn}\} . \tag{3.12}$$

By using Eqs. (2.9), (2.10), and (2.17), Eq. (3.11) can be transformed into

$$\hat{G}^{(j)}(\omega,\mathbf{R})=(\omega/c)^2\frac{\sigma^{(j)}_m}{4\pi R}\frac{|K^{(j)}_{Gm}|^{1/2}}{m^{(j)}} \times\hat{\mathscr{E}}^{(j)}\exp\{i(\omega/c)(1/m^{(j)})R\} , \tag{3.13}$$

where $\sigma^{(j)}_m=\exp\{(i/2)\arg K^{(j)}_{Gm}\}$ and $m^{(j)}$, $\mathbf{e}^{(j)}$ are defined by Eqs. (2.13) and (2.15). The calculation of $K^{(j)}_{Gm}$ is given in the Appendix. It has been taken into account in Eq. (3.13) that the sign of $K^{(j)}_{Gn}$ is equal to the sign of $K^{(j)}_{Gm}$ in the corresponding points, i.e., the equality $\sigma^{(j)}_n=\sigma^{(j)}_m$ is valid.

Equation (3.13) is general and as applicable for isotropic media as for uniaxial and biaxial anisotropic media. [In isotropic and uniaxial media, it is necessary to modify Eqs. (2.7) and (2.15) for polarizations of normal waves in these media.] In the isotropic and uniaxial cases, there is one stationary point on each of the surfaces $A^{+}_{(j)}$ from Eq. (3.5). The situation is more complicated in the biaxial case. If the vector $\mathbf{r}$ is directed out of the internal conic refraction cone, there is one stationary point on each part of the wave surface, as before [Fig. 1(a)]. In this case, $\sigma^{(1)}_m=\sigma^{(2)}_m=1$. If $\mathbf{r}$ coincides with a generator of this cone, one stationary point remains on the external part of the wave surface ($\sigma^{(1)}_m=1$), and the other coincides with the singular point of the wave surface, which is common for its internal and external parts [Fig. 1(b)]. If, at last, $\mathbf{r}$ is directed into this cone but does not coincide with a biradial, both stationary points are situated on the external part of the wave surface, and there are no stationary points on its internal part [Fig. 1(c)]. In this case $\sigma^{(1)}_m=1$ and $\sigma^{(2)}_m=i$.

When the vector $\mathbf{r}$ is directed along a biradial, Eq. (3.8) becomes degenerate: one direction of $\mathbf{r}$ corresponds to an infinite set of wave vectors generated in the internal conic refraction cone [see Fig. 1(d); OA and OB are two generators of this cone situated on the plane of the figure]. Equations (3.10) and (3.13) are inapplicable in this case. Formally this fact is exhibited in the ray-surface Gaussian curvature approaching infinity. It indicates that the Green's function $\hat{G}(\omega,\mathbf{R})$ decreases more slowly than

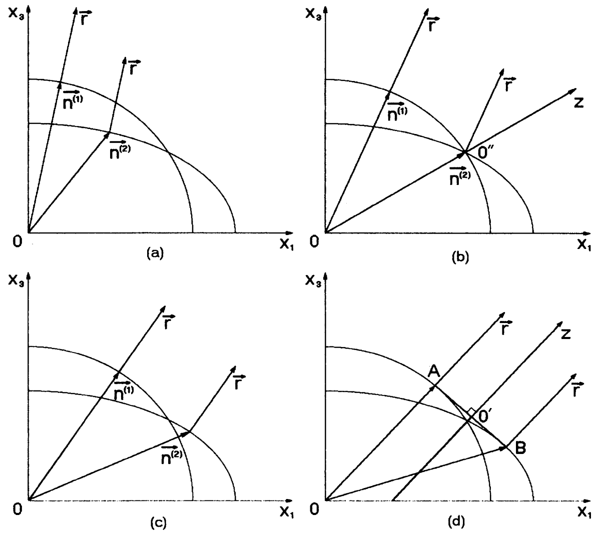

FIG. 1. Arrangement of the stationary points $\mathbf{k}^{(j)}$ on the internal and external parts of the wave surfaces dependent on the direction $\mathbf{r}$. The wave surface is given in the section by the plane containing singular points. (a) $\mathbf{r}$ is directed out from internal conic refraction cone. (b) $\mathbf{r}$ is directed along one of determinators of this cone. (c) $\mathbf{r}$ is directed into this cone. (d) $\mathbf{r}$ is directed along biradial.

R^{-1} along this direction.

In the case of the vector $\mathbf{r}$ coinciding with a generator of the internal conic refraction cone, the asymptotic behavior of the term with $j=2$ in Eq. (3.10) also differs from R^{-1}. This difference is exhibited in the ray-surface Gaussian curvature approaching zero in the corresponding points of this surface. In this case, $\hat{G}^{(2)}(\omega,\mathbf{R})$ decreases more quickly than R^{-1}.

In Secs. IV and V calculations of the Green's-function asymptotic behavior are carried out for these two special cases.

IV. THE CASE OF NEARLY BIRADIAL RAY VECTORS

The direction $\mathbf{r}$ coinciding with a biradial (we designate it as $\mathbf{r}_{\rm bir}$) corresponds to a ray-surface singular point. On the wave surface, the whole circle Ω of stationary points corresponds to this kind of ray-vector direction.

The standard approach to the phase-integral calculation in the situation with the whole line of stationary points $L_{\rm st}$ is to use the following variables in the vicinity of $L_{\rm st}$: the longitudinal variable, which characterizes the shift along $L_{\rm st}$, and the transverse variables, which characterized the shift across $L_{\rm st}$. Integrals over the transverse variables are calculated by the ordinary stationary-phase method, and, in the rest integral over the longitudinal variable, there is no longer any large parameter (i.e., it is a numerical factor).

In our case, for the wave-surface description, it is convenient to pass to the cylindrical coordinate system (ρ,ϕ,z), in which the origin is situated in the center of the circle Ω, and the axis is directed along the biradial. We choose the cylindrical coordinate system connected to the Cartesian system of the tensor $\hat{\varepsilon}$'s principal axes (x_1,x_2,x_3) by the relationships

$$\begin{aligned}\rho\cos\phi&=x_1\cos\psi_1-x_3\sin\psi_1-\rho_1\,,\\ \rho\sin\phi&=x_2\,,\\ z&=x_1\sin\psi_1+x_3\cos\psi_1-\varepsilon_2^{1/2}\,,\end{aligned}\tag{4.1}$$

where

$$\begin{aligned}\tan\psi_1&=\left[\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_3-\varepsilon_2}\right]^{1/2}\,,\\ \rho_1&=\tfrac{1}{2}\left[\frac{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}{\varepsilon_2}\right]^{1/2}\,,\end{aligned}\tag{4.2}$$

$0\le\psi\le\pi/2$. For definiteness, we set here and throughout that $\varepsilon_1<\varepsilon_2<\varepsilon_3$. Then the equation of the circle Ω takes the form: $\rho=\rho_1$, $z=0$. This situation is shown in Fig. 1(d). The axis corresponding to the eigenvalue ε_2 and the plane of the circle Ω is perpendicular to the plane of the figure. The value $AB=2\rho_1$ is a diameter of the circle Ω, and the point O' is its center.

By rewriting Fresnel's equation (2.3) in the Cartesian coordinates connected with tensor $\hat{\varepsilon}$ principal axes and taking into account Eq. (4.1), we can obtain that the equation of wave surface takes the form

$$\begin{aligned}&[\rho^2+2\rho\rho_1\cos\phi+\rho_1^2+z^2+2\varepsilon_2^{1/2}z][\varepsilon_2(\rho^2+2\rho\rho_1\cos\phi+\rho_1^2)+(\varepsilon_1+\varepsilon_3-\varepsilon_2)(z^2+2\varepsilon_2^{1/2}z)\\ &\qquad-4\varepsilon_2^{1/2}\rho_1(\rho\cos\phi+\rho_1)(z+\varepsilon_2^{1/2})+4\varepsilon_2\rho_1^2]-4\varepsilon_2\rho^2\rho_1^2\sin^2\phi=0\,.\end{aligned}\tag{4.3}$$

Substituting $\rho=\rho_1$ and $z=0$ into Eq. (4.3), we find that this equation becomes an identity. It means that the circle Ω is really situated on the wave surface. It is convenient to choose the following parametrization for the wave surface:

$$z=z(\rho,\phi)\,.\tag{4.4}$$

By using the surface equation (4.4) in the unexplicit form (4.3), it is not difficult to calculate the partial derivatives of the function $z(\rho,\phi)$ with respect to the variables ρ and ϕ up to second order at points of the circle Ω,

$$z_\rho(\rho_1,\phi)=z_\phi(\rho_1,\phi)=z_{\rho\phi}(\rho_1,\phi)=z_{\phi\phi}(\rho_1,\phi)=0\ ,$$
$$z_{\rho\rho}(\rho_1,\phi)=-\varepsilon_2^{3/2}[\varepsilon_2\sin^2(\phi/2)+\varepsilon_1\cos^2(\phi/2)]^{-1}[\varepsilon_2\sin^2(\phi/2)+\varepsilon_3\cos^2(\phi/2)]^{-1}\ . \tag{4.5}$$

Actually, the fact that the first derivatives in Eq. (4.5) equal zero means that the wave-surface normals at every point of the circle Ω have a common direction (along the z axis). The second derivatives $z_{\phi\phi}$ and $z_{\rho\phi}$ (together with z_ρ and z_ϕ) equaling zero means that the wave-surface Gaussian curvature in the points of circles Ω is equal to zero. This fact can be seen from the relationship

$$K_G(\rho,\phi)=\frac{\rho^2 z_{\rho\rho}(\rho z_\rho+z_{\phi\phi})-(\rho z_{\rho\phi}-z_\phi)^2}{(\rho^2+\rho^2 z_\rho^2+z_\phi^2)^2}\ , \tag{4.6}$$

which is the consequence of Eq. (A7).

The integral (3.7) in terms of the variables ρ and ϕ takes the form

$$\hat{G}(\omega,\mathbf{R}_{\rm bir})=(\omega/c)^3\frac{\exp\{i\pi/2\}\exp\{i(\omega/c)\varepsilon_2^{1/2}R\}}{8\pi^2}\times\sum_{j=1,2}\int_{A^+_{(j)}}\rho\,d\rho\,d\phi\,\hat{\mathscr{E}}_{(j)}(\rho,\phi)n_{(j)}(\rho,\phi)\cos\delta_{(j)}(\rho,\phi)\exp\{i(\omega/c)z(\rho,\phi)R\}\ . \tag{4.7}$$

Let us calculate this integral over the transverse variable ρ (the variable ϕ is fixed) by the stationary-phase method. There is no contribution in the main term of the asymptotic expansion from the term $j=2$ in Eq. (4.7), because the circle Ω is situated on the external ($j=1$) part of the wave surface. Thus taking into account that at every point of the circle Ω $n_{(1)}\cos\delta_{(1)}=\varepsilon_2^{1/2}$, $(\mathbf{e}_{(1)}\hat{\varepsilon}\mathbf{e}_{(1)})=\varepsilon_2$, we obtain

$$\hat{G}(\omega,\mathbf{R}_{\rm bir})=(\omega/c)^{5/2}R^{-1/2}\frac{\exp\{i\pi/4\}\exp\{i(\omega/c)\varepsilon_2^{1/2}R\}}{2(2\pi)^{3/2}}\hat{F}\ , \tag{4.8}$$

where

$$\hat{F}=\frac{\rho_1}{\varepsilon_2^{1/2}}\int_0^{2\pi}d\phi\frac{\mathbf{e}_{(1)}(\rho_1,\phi)\otimes\mathbf{e}_{(1)}(\rho_1,\phi)}{|z_{\rho\rho}(\rho_1,\phi)|^{1/2}}\ . \tag{4.9}$$

The vector $\mathbf{e}_{(1)}(\rho_1,\phi)$ can be written in the form

$$\mathbf{e}_{(1)}(\rho_1,\phi)=\sin(\phi/2)\mathbf{a}+\cos(\phi/2)\mathbf{b}\ , \tag{4.10}$$

where $\mathbf{a}$ is the unit vector directed along the principal axis of the tensor $\hat{\varepsilon}$ corresponding to the eigenvalue ε_2 and $\mathbf{b}=\mathbf{r}\times\mathbf{a}$. Thus Eq. (4.9) can be rewritten in the form

$$\hat{F}=(\rho_1/\varepsilon_2^{5/4})(\mathbf{a}\otimes\mathbf{a}I_1+\mathbf{b}\otimes\mathbf{b}I_2)\ , \tag{4.11}$$

where the integrals

$$I_1=\int_0^{2\pi}d\phi\,\sin^2(\phi/2)[\varepsilon_2\sin^2(\phi/2)+\varepsilon_1\cos^2(\phi/2)]^{1/2}[\varepsilon_2\sin^2(\phi/2)+\varepsilon_3\cos^2(\phi/2)]^{1/2}\ ,$$
$$I_2=\int_0^{2\pi}d\phi\,\cos^2(\phi/2)[\varepsilon_2\sin^2(\phi/2)+\varepsilon_1\cos^2(\phi/2)]^{1/2}[\varepsilon_2\sin^2(\phi/2)+\varepsilon_3\cos^2(\phi/2)]^{1/2} \tag{4.12}$$

can be represented, if it is necessary, as a sum of an elliptic integrals. Thus for the beam propagating along the biradial direction, the Green's function asymptotically decreases as $R^{-1/2}$.

For the description of the smooth transition from the asymptotic form proportional to $R^{-1/2}$ for $\mathbf{r}=\mathbf{r}_{\rm bir}$ to the asymptotic form proportional to R^{-1} for other directions, we make use of the asymptotic representation that is uniform with respect to the parameter characterizing the deflection from the biradial.

Let us apply the stationary-phase method for the case of $\mathbf{r}$ noncoincident with $\mathbf{r}_{\rm bir}$ in the form similar to the case of $\mathbf{r}=\mathbf{r}_{\rm bir}$. Carrying out the first integration over the variable ρ (the variable ϕ is fixed), we obtain that the stationary points become a function of the angle ϕ: $\rho_{\rm st}=\rho_{\rm st}(\phi)$. It leads to the integral over the variable ϕ dependence on the large parameter $(\omega/c)R$. (This dependence disappears in the limit $\mathbf{r}\to\mathbf{r}_{\rm bir}$.) The integral over the variable ϕ, as a function of the large parameter $(\omega/c)R$, gives us the uniform asymptotic form we are interested in.

Let us confine our examination to the directions that are deflected from the biradial on the small angle θ_1 only. For convenience, let us represent vectors in the Cartesian coordinate system connected with the cylindrical coordinate system (4.1) by the relationships

$$x=\rho\cos\phi\ ,\quad y=\rho\sin\phi\ ,\quad z=z\ . \tag{4.13}$$

The linear approximation over θ_1 gives us

$$\mathbf{r}=(\theta_1\cos\phi_1,\theta_1\sin\phi_1,1) , \tag{4.14}$$

where the angle ϕ_1 characterizes a deflection direction of the vector $\mathbf{r}$ in the plane perpendicular to the axis z. If the parameter θ_1 is rather small, the points $\rho_{st}(\phi)$ are situated in the vicinity of the circle Ω. That is why the quadratic approximation over the parameter $\Delta=\rho-\rho_1$ is rather good for the description of the wave surface $z=z(\rho,\phi)$,

$$z(\rho,\phi)=\tfrac{1}{2}z_{\rho\rho}(\rho_1,\phi)\Delta^2 . \tag{4.15}$$

In this approximation, the vector $\mathbf{n}$, as a function of the variables Δ and ϕ, can be written in the form

$$\mathbf{n}(\Delta,\phi)=(\rho_1+(\rho_1+\Delta)\cos\phi,(\rho_1+\Delta)\sin\phi\ ,\ \varepsilon_2^{1/2}+\tfrac{1}{2}z_{\rho\rho}(\rho_1,\phi)\Delta^2) . \tag{4.16}$$

By using Eqs. (4.14) and (4.16), we find that the exponent term for $j=1$ in Eq. (3.7) takes the form

$$i(\omega/c)[\rho_1\theta_1\cos\phi_1+(\rho_1+\Delta)\theta_1\cos(\phi-\phi_1) +\varepsilon_2^{1/2}+\tfrac{1}{2}z_{\rho\rho}(\rho_1,\phi)\Delta^2]R . \tag{4.17}$$

Let us carry out the integration by a stationary-phase method over the variable Δ. In this way, the value

$$\Delta_{st}=-\frac{\theta_1\cos(\phi-\phi_1)}{z_{\rho\rho}(\rho_1,\phi)} \tag{4.18}$$

is the stationary point. Thus we obtain

$$\hat{G}(\omega,\mathbf{R})=(\omega/c)^{5/2}\frac{\exp\{i\pi/4\}\varepsilon_2^{1/2}\rho_1}{2(2\pi)^{3/2}R^{1/2}}\exp\{i(\omega/c)\varepsilon_2^{1/2}R\}\exp\{i(\omega/c)\rho_1\cos\phi_1\theta_1R\} \times\int_0^{2\pi}d\phi\frac{\hat{\mathscr{E}}_{(1)}(\rho_1,\phi)}{|z_{\rho\rho}(\rho_1,\phi)|^{1/2}}\exp\{i(\omega/c)\rho_1\cos(\phi-\phi_1)\theta_1R\} . \tag{4.19}$$

This equality gives the asymptotic representation that is uniform with respect to the parameter $(\omega/c)\theta_1R$, if the inequalities $\theta_1\ll1,(\omega/c)\theta_1^2R\ll1$ are valid. If the last inequality is not valid but $\theta_1\ll1$, Eq. (4.19) becomes not strictly asymptotical but still remains a rather good approximation. In the limit $(\omega/c)\theta_1R\to0$, the exponent in the integral over the variable ϕ tends to zero and Eq. (4.19) turns into Eq. (4.8) giving an asymptotic form proportional to $R^{-1/2}$. If $(\omega/c)\theta_1R\gg1$, the integral over the variable ϕ in Eq. (4.19) can be calculated by a stationary-phase method. In this case, the stationary points are $\phi_{st}^{(1)}=\phi_1$ and $\phi_{st}^{(2)}=\phi_1+\pi$, and we obtain for the Green's-function asymptotic form

$$\hat{G}(\omega,\mathbf{R})=(\omega/c)^2\frac{\varepsilon_2^{1/2}\rho_1^{1/2}\exp\{i(\omega/c)\varepsilon_2^{1/2}R\}}{4\pi R\,\theta_1^{1/2}}\sum_{j=1,2}\sigma^{(j)}\frac{\hat{\mathscr{E}}_{(1)}(\rho_1,\phi^{(j)})}{|z_{\rho\rho}(\rho_1,\phi^{(j)})|^{1/2}}\exp\{i(\omega/c)\rho_1[\cos\phi_1-(-1)^j]\theta_1R\} , \tag{4.20}$$

where $\sigma^{(1)}=1$, $\sigma^{(2)}=i$. Equation (4.20) is equivalent to Eq. (3.10), if we consider the deflection of the vector $\mathbf{r}$ from the biradial to be small. The divergence of Eq. (4.20) for $\theta_1\to0$ is connected with the Gaussian curvature of the ray surface approaching infinity in Eq. (3.12) in the limit $\mathbf{r}\to\mathbf{r}_{bir}$. The terms $j=1$ and $j=2$ in Eq. (4.20) correspond to the two types of the extraordinary waves that can propagate in a biaxial medium along the direction $\mathbf{r}$. Note that both terms $j=1,2$ are connected with the contribution into $\hat{G}(\omega,\mathbf{R})$ from the external part of the wave surface [$j=1$ in Eq. (3.7)].

V. THE CASE OF THE WAVE VECTOR DIRECTED ALONG THE BINORMAL

For directions $\mathbf{r}$ coinciding with the generators of the internal conic refraction cone, the Gaussian curvature of the wave surface for the term $j=2$ in Eq. (3.10) turns into infinity. It is connected with the fact that the wave vector $\mathbf{k}^{(2)}(\mathbf{r})$ corresponding to this choice of vector $\mathbf{r}$ is directed along a binormal and reaches the singular point of the wave surface which is common for its external and internal parts. The calculation of the contribution into the Green's function $\hat{G}^{(2)}(\omega,\mathbf{R})$ corresponding to this point requires a special examination. At the same time, the wave vector $\mathbf{k}^{(1)}(\mathbf{r})$ corresponds to the nonsingular point on the wave surface, in which Gaussian curvature is finite, and the corresponding contribution $\hat{G}^{(1)}(\omega,\mathbf{R})$ can be found from Eq. (3.13) (it is the term with $j=1$).

The integral (3.7) that defines the asymptotic form of $\hat{G}^{(2)}(\omega,\mathbf{R})$ is not quite usual for the stationary-phase method in this case. First, as we have noted above, the stationary point for $\hat{G}^{(2)}(\omega,\mathbf{R})$ is situated at the singular point of the wave surface. Second, the stationary point is placed on a boundary of the integration regions for the terms $j=1$ and $j=2$ in Eq. (3.7). According to general principles of the stationary-phase method, the principal contribution to the phase integral in this situation is the contribution from the vicinity of the singular point. For calculation purposes, it is sufficient to keep the principal terms of the asymptotic expansions of the phase function and the nonexponential factor. Expanding the integration limits to infinity far from the singular-point zone, we obtain the so-called canonical integral. This integral defines the asymptotic form we are interested in.

For convenience of the calculation, let us use the cylindrical coordinate system (ρ,ϕ,z) with the origin O'' situated in the singular point of the wave surface and the axis directed along the binormal [Fig. 1(b)]:

$$\begin{aligned}\rho\cos\phi&=-x_1\sin\psi_2+x_3\cos\psi_2\ ,\\ \rho\sin\phi&=x_2\ ,\\ z&=x_1\cos\psi_2+x_3\sin\psi_2-\varepsilon_2^{1/2}\ ,\end{aligned}\tag{5.1}$$

where

$$\tan\psi_2=\left[\frac{\varepsilon_1(\varepsilon_3-\varepsilon_2)}{\varepsilon_3(\varepsilon_2-\varepsilon_1)}\right]^{1/2}\ ,\tag{5.2}$$

$0\le\psi_2\le\pi/2$. In terms of the variables (ρ,ϕ,z), the wave-surface equation (2.3) takes the form

$$[z^2+2\varepsilon_2^{1/2}z+\rho^2][\varepsilon_1\varepsilon_3\varepsilon_2^{-1}(z^2+2\varepsilon_2^{1/2}z)+\varepsilon_2\rho^2\sin^2\phi+(\varepsilon_1+\varepsilon_3-\varepsilon_1\varepsilon_3\varepsilon_2^{-1})\rho^2\cos^2\phi +2\varepsilon_2^{-1}\varepsilon_1^{1/2}\varepsilon_3^{1/2}(\varepsilon_3-\varepsilon_2)^{1/2}(\varepsilon_2-\varepsilon_1)^{1/2}(z+\varepsilon_2^{1/2})\rho\cos\phi]-(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)\rho^2\sin^2\phi=0\ .\tag{5.3}$$

Let us choose the parametrization of the wave surface by analogy with Sec. IV: $z=z(\rho,\phi)$. From Eq. (5.3) it is not difficult to find the partial derivatives of $z(\rho,\phi)$ with respect to ρ up to second order at the singular point $(\rho=0,z=0)$ on the external and internal parts of the wave surface.

$$\begin{aligned}z_{(j)\rho}(0,\phi)&=A[(-1)^{j+1}-\cos\phi]\ ,\\ z_{(j)\rho\rho}(0,\phi)&=-2B[1-(-1)^jD\cos\phi][1+(-1)^jE\cos\phi]\ ,\end{aligned}\tag{5.4}$$

where

$$\begin{aligned}A&=\frac{1}{2}\left[\frac{(\varepsilon_3-\varepsilon_2)(\varepsilon_2-\varepsilon_1)}{\varepsilon_1\varepsilon_3}\right]^{1/2}\ ,\\ B&=\frac{(\varepsilon_3+\varepsilon_2)(\varepsilon_2+\varepsilon_1)}{8\varepsilon_1\varepsilon_3\varepsilon_2^{1/2}}\ ,\\ D&=\frac{\varepsilon_3-\varepsilon_2}{\varepsilon_2+\varepsilon_3}\ ,\quad E=\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\ ,\end{aligned}\tag{5.5}$$

and the subscript j $(j=1,2)$ refers to the external and internal parts of the wave surface. The dependence on the angle ϕ in Eq. (5.4) shows that the point $(\rho=0,z=0)$ is really singular. Taking into account Eq. (5.4), we can write the expansion into series of $z_{(j)}(\rho,\phi)$ in the vicinity of the singular point up to the second-order approximation,

$$z_{(j)}(\rho,\phi)=A[(-1)^{j+1}-\cos\phi]\rho-\beta_{(j)}(\phi)\rho^2\ ,\tag{5.6}$$

where

$$\beta_{(j)}(\phi)=-\tfrac{1}{2}z_{(j)\rho\rho}(0,\phi)\ .\tag{5.7}$$

By using Eq. (5.6), it is easy to see, specifically, that the Gaussian curvature of the wave surface is really increasing without bounds in the the vicinity of the point $\rho=0$. Actually, for every angle ϕ the numerator of Eq. (4.6) is proportional to ρ^3, and the denominator is proportional to ρ^4.

It follows from Eq. (5.6) that in the vicinity of the singular point, where we can neglect the term proportional to ρ^2, the wave surface $z_{(j)}(\rho,\phi)$ $(j=1,2)$ is a cone. The total combination of normals to this cone at its vertex is the internal conic refraction cone. Let us characterize vector $\mathbf{r}$ by the values (ρ_0,ϕ_0,z_0). For directions $\mathbf{r}$ coinciding with the generators of the internal conic refraction cone, we obtain

$$\begin{aligned}\rho_0&=2Az_0\cos\phi_0\ ,\\ z_0&=(1+4A^2\cos^2\phi_0)^{-1/2}\ .\end{aligned}\tag{5.8}$$

Then Eq. (3.7) yields for the contribution $\hat{G}^{(2)}(\omega,\mathbf{R})$,

$$\hat{G}^{(2)}(\omega,\mathbf{R})=(\omega/c)^3\frac{\exp\{i\pi/2\}\exp\{i(\omega/c)\varepsilon_2^{1/2}z_0R\}}{8\pi^2}\times\sum_{j=1,2}\int_{A_{(j)}^{+}}\rho\,d\rho\,d\phi\,\hat{\mathscr{E}}_{(j)}(\rho,\phi)n_{(j)}(\rho,\phi)\cos\delta_{(j)}(\rho,\phi)\exp\{i(\omega/c)z_0g_{(j)}(\rho,\phi)R\}\ ,\tag{5.9}$$

after taking into account Eqs. (5.6) and (5.8). Here

$$g_{(j)}(\rho,\phi)=A[(-1)^{j+1}+\cos(\phi-2\phi_0)]\rho-\beta_{(j)}(\phi)\rho^2\ .\tag{5.10}$$

Setting the first derivatives of $g_{(j)}(\rho,\phi)$ with respect to ρ and ϕ equal to zero, we find the stationary points in the vicinity of the singular point:

$$\rho_{\mathrm{st}(j)}=0\ ,\quad \phi_{\mathrm{st}(j)}=2\phi_0+(2-j)\pi\ .\tag{5.11}$$

Expanding the functions $g_{(j)}(\rho,\phi)$ into series in the vicinity of the stationary points (5.11), keeping the first nonzero terms, and introducing the variables $\tilde\rho=\rho-\rho_{st}$ and $\tilde\phi=\phi-\phi_{st}$ we find that the problem is reduced to the calculation of the canonical integrals of the following type:

$$\int_0^\infty d\tilde\rho\int_{-\infty}^{\infty}d\tilde\phi\,\tilde\rho\exp\{i(\omega/c)(C_1\tilde\rho+C_2\tilde\rho\tilde\phi^2)R\}\ , \qquad (5.12)$$

where C_1 and C_2 are constants. The singularity of the stationary point in terms of the variables ρ and ϕ is reflected in the existence of the term $\tilde\rho\tilde\phi^2$ in the exponent. Note that the inferior limit in the integral over the variable $\tilde\rho$ in Eq. (5.12) is equal to zero. It is connected with the fact that the stationary point is situated on the boundary of the integration region.

It is easy to calculate the integral (5.12) integrated first over the variable $\tilde\phi$ and then over the variable $\tilde\rho$. After adding up the external and the internal parts of the wave surface ($j=1,2$), we obtain

$$\hat G^{(2)}(\omega,\mathbf{R})=(\omega/c)^{7/4}R^{-5/4}\frac{\exp\{i\pi/8\}\Gamma(3/4)}{8\pi^{3/2}}\frac{\exp\{i(\omega/c)\varepsilon_2^{1/2}z_0R\}}{\varepsilon_2^{1/2}A^{1/2}z_0^{9/4}\beta_{(2)}^{3/4}(2\phi_0)}\mathbf{e}^{(2)}(\mathbf{r})\otimes\mathbf{e}^{(2)}(\mathbf{r})\ , \qquad (5.13)$$

where $\Gamma(x)$ is the gamma function and

$$\mathbf{e}^{(2)}(\mathbf{r})=z_0\left[-\left[\frac{\varepsilon_2(\varepsilon_3-\varepsilon_2)}{\varepsilon_1(\varepsilon_3-\varepsilon_1)}\right]^{1/2}\cos\phi_0\ ,\ \ \sin\phi_0\ ,\ \left[\frac{\varepsilon_2(\varepsilon_2-\varepsilon_1)}{\varepsilon_3(\varepsilon_3-\varepsilon_1)}\right]^{1/2}\cos\phi_0\right] \qquad (5.14)$$

expressed in the principal axes of the tensor $\hat\varepsilon$.

The asymptotic decrease of the contribution $\hat G^{(2)}(\omega,\mathbf{R})$ is proportional to $R^{-5/4}$, i.e., this contribution decreases more quickly than the contribution $\hat G^{(1)}(\omega,\mathbf{R})$ which is proportional to R^{-1}. Thus for $\mathbf{r}$ directed along the generators of the internal conic refraction cone, two types of the waves possessing different asymptotic behavior are possible.

VI. CONCLUSION

Thus the Green's function in biaxial media is rather complicated. In almost all directions, it can be written as a sum of these waves [Eqs. (3.10) and (3.13)], that asymptotically decreases as R^{-1}. The polarization vectors of these waves are perpendicular to each other. For $\mathbf{r}$ directed along the generators of the internal conic refraction cone, there are also two waves with perpendicular polarization vectors. The wave of one of the polarizations possesses the usual asymptotic form proportional to R^{-1} [the term $j=1$ in Eq. (3.13)], and that of the other polarization possesses the asymptotic form proportional to $R^{-5/4}$ [Eq. (5.13)]. If $\mathbf{r}$ is directed along a biradial, the Green's function asymptotically decreases as $R^{-1/2}$. In this case, the wave takes a form of one wave of the linear polarization [Eqs. (4.8) and (4.11)]. This polarization results from adding up an infinite set of the waves with different polarizations but equal phase factors [the integration over the variable ϕ in Eq. (4.9)]. Therefore, we can conclude that the polarization properties of the wave propagating along a biradial direction in a biaxial medium is similar to those in an isotropic medium. If we regard a point dipole as a source of the radiation, we can obtain every preliminarily given polarization through its rotation. Note that we can speak about polarizations for

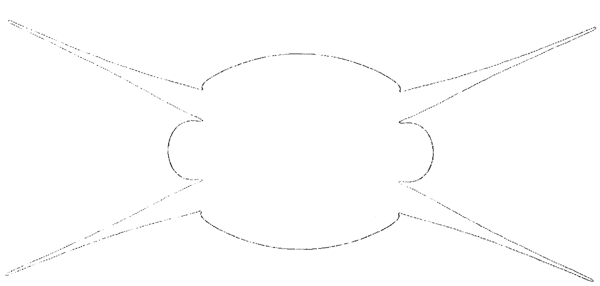

FIG. 2. Angle dependence of $\ln[1+|\mathrm{Tr}\hat G(\omega,\mathbf{R})|]$ in the section by the plane containing biradials.

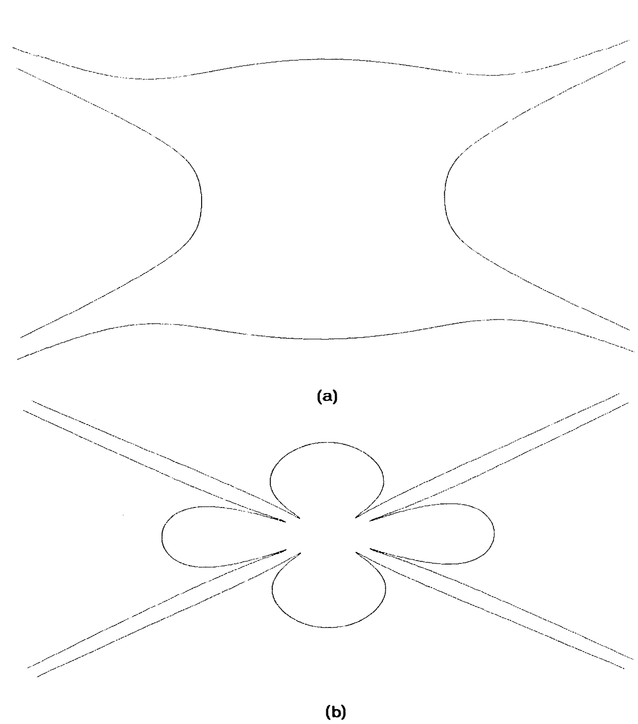

FIG. 3. Angle dependence of (a) $|\mathrm{Tr}\hat G^{(1)}(\omega,\mathbf{R})|$ and (b) $|\mathrm{Tr}\hat G^{(2)}(\omega,\mathbf{R})|$ in the section by the plane containing biradials.

the special directions only in the region where it is possible to consider this wave to be a plane wave.

For illustration of the Green's-function dependence on the direction **r**, it is convenient to regard the scalar quantity $|\mathrm{Tr}\hat{G}(\omega,\mathbf{R})|$. The angle dependence of $|\mathrm{Tr}\hat{G}(\omega,\mathbf{R})|$ is shown in Fig. 2 in the section of the plane containing biradials. The sharp maximums correspond to the transitions to the asymptotic form proportional to $R^{-1/2}$.

For all directions except that in the vicinity of biradials, the Green's function may be written as a sum of two contributions from the external and internal parts of the wave surface: $\hat{G}^{(1)}(\omega,\mathbf{R})$ and $\hat{G}^{(2)}(\omega,\mathbf{R})$. Figure 3 illustrates the angle dependence of the scalar quantities $|\mathrm{Tr}\hat{G}^{(1)}(\omega,\mathbf{R})|$ and $|\mathrm{Tr}\hat{G}^{(2)}(\omega,\mathbf{R})|$ in the same section as in Fig. 2. The minima in Fig. 3(b) correspond to the transitions to the asymptotic form proportional to $R^{-5/4}$.

If we apply Eqs. (3.10) and (3.13) to all directions, the values $|\mathrm{Tr}\hat{G}^{(2)}(\omega,\mathbf{R})|$ in the points of the minimum in Fig. 3(b) would be equal to zero, and the values $|\mathrm{Tr}\hat{G}(\omega,\mathbf{R})|$ in the points of the sharp maximum vicinity in Fig. 2 would increase infinitely.

Note that the field decreasing more slowly than R^{-1} does not violate the energy conservation law, because the width of the peak corresponding to this asymptotic behavior decreases, if the value R increases.

We emphasize that the results obtained in this paper are related to the case of sources and receivers situated inside a biaxial medium. For light-scattering problems, the receiver is usually situated outside the medium, and it is necessary to take into account some corrections connected with the refraction of the light on the boundary of the medium [3]. In particular for single scattering, these refractive corrections cancel the Gaussian curvatures in Eqs. (3.11) and (3.13) exactly. It leads to the disappearance of the unusual asymptotic behavior obtained in Secs. IV and V. However, if we carry out our measurements near the boundary of a biaxial medium (for example, on this boundary), the effect of the unusual asymptotic behavior of the field can be observed. Moreover, the relationship for the Green's function inside the medium without the refraction corrections is necessary for examination of multiple scattering. In the case of a biaxial medium, the unusual asymptotic behavior discussed above can lead to intensification of multiple scattering in the special geometries of the experiment.

APPENDIX: GAUSSIAN CURVATURES OF THE WAVE SURFACE AND THE RAY SURFACE

For a plane wave in a medium characterized by the vectors **n** and **m**, we can write from Gaussian *theorema egregium* [8]

$$\begin{aligned} d\Omega_m &= K_{Gn}\, dA_n \ , \\ d\Omega_n &= K_{Gm}\, dA_m \ . \end{aligned} \tag{A1}$$

Here dA_n and dA_m are the infinitesimal area elements of the wave surface and the ray surface in the vicinity of the points defined by the vectors **n** and **m**, and $d\Omega_m$, $d\Omega_n$ are the infinitesimal solid angles which are determined by the total of normals to area elements defined above. The area elements dA_n and dA_m are considered to be connected: the ray vectors that determine the area dA_m correspond to the vectors **n** determined the area dA_n. Occasionally Eqs. (A1) have been taken as the definition of a Gaussian curvature [11]. Since a normal to dA_m is directed along **n** and a normal to dA_n is directed along **m**, we can write

$$\begin{aligned} d\Omega_m &= m^{-2}\cos\delta\, dA_m \ , \\ d\Omega_n &= n^{-2}\cos\delta\, dA_n \ . \end{aligned} \tag{A2}$$

By substituting Eq. (A2) into Eq. (A1) and taking into account Eq. (2.10), we obtain

$$K_{Gn}K_{Gm} = \cos^4\delta \ . \tag{A3}$$

It is easy to rewrite Eq. (2.12) in the form

$$\begin{aligned} &(m_1^2+m_2^2+m_3^2)(\varepsilon_1^{-1}m_1^2+\varepsilon_2^{-1}m_2^2+\varepsilon_3^{-1}m_3^2)-[\varepsilon_1^{-1}(\varepsilon_2^{-1}+\varepsilon_3^{-1})m_1^2 \\ &\qquad +\varepsilon_2^{-1}(\varepsilon_1^{-1}+\varepsilon_3^{-1})m_2^2+\varepsilon_3^{-1}(\varepsilon_1^{-1}+\varepsilon_2^{-1})m_3^2]+\varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_3^{-1}=0 \ , \end{aligned} \tag{A4}$$

where m_1, m_2, and m_3 are projections of the vector **m** at the principle axes of the tensor $\hat{\varepsilon}$, and we consider $\varepsilon_1<\varepsilon_2<\varepsilon_3$. By taking into account the ray-surface symmetry, we can confine our examination to the region $m_1>0$, $m_2>0$, $m_3>0$. From Eq. (A4) we can obtain m_3 as a function of m_1, m_2:

$$\begin{aligned} m_3^{(j)} &= f^{(j)}(m_1,m_2) \\ &= [\tfrac{1}{2}\varepsilon_3(-\beta(m_1,m_2)+(-1)^j\Delta^{1/2}(m_1,m_2))]^{1/2} \ . \end{aligned} \tag{A5}$$

Here

$$\begin{aligned} &\beta(m_1,m_2)=(\varepsilon_1^{-1}+\varepsilon_3^{-1})m_1^2+(\varepsilon_2^{-1}+\varepsilon_3^{-1})m_2^2-\varepsilon_3^{-1}(\varepsilon_1^{-1}+\varepsilon_2^{-1}) \ , \\ &\Delta(m_1,m_2)=\beta^2(m_1,m_2)-4\varepsilon_3^{-1}\gamma(m_1,m_2) \ , \\ &\gamma(m_1,m_2)=(m_1^2+m_2^2)(\varepsilon_1^{-1}m_1^2+\varepsilon_2^{-1}m_2^2)+\varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_3^{-1}-\varepsilon_1^{-1}(\varepsilon_2^{-1}+\varepsilon_3^{-1})m_1^2-\varepsilon_2^{-1}(\varepsilon_1^{-1}+\varepsilon_3^{-1})m_2^2 \ , \end{aligned} \tag{A6}$$

and the superscript j $(j=1,2)$ corresponds to the internal and external parts of the ray surface.

In these terms, the Gaussian curvature of the ray surface can be found from the relationship [8]

$$K_{Gm}^{(j)}=\frac{f_{11}^{(j)}f_{22}^{(j)}-f_{12}^{(j)^2}}{(1+f_1^{(j)^2}+f_2^{(j)^2})^2}\ , \tag{A7}$$

where the subscripts 1 and 2 signify the partial derivatives of $f^{(j)}(m_1,m_2)$ to m_1 and m_2. These derivatives are defined by

$$\begin{aligned}
f_i^{(j)}&=\frac{-2\Delta^{1/2}\beta_i+(-1)^j\Delta_i}{8\varepsilon_3^{-1}\Delta^{1/2}f^{(j)}}\ ,\\
f_{ii}^{(j)}&=\frac{-4\Delta^{3/2}\beta_{ii}+(-1)^j(2\Delta_{ii}\Delta-\Delta_i^2)}{16\varepsilon_3^{-1}\Delta^{3/2}f^{(j)}}-\frac{(2\Delta^{1/2}\beta_i-(-1)^j\Delta_i)^2}{64\varepsilon_3^{-2}\Delta f^{(j)^3}}\ ,\\
f_{12}^{(j)}&=(-1)^j\frac{2\Delta_{12}\Delta-\Delta_1\Delta_2}{16\varepsilon_3^{-1}\Delta^{3/2}f^{(j)}}-\frac{\prod_{i=1,2}(2\Delta^{1/2}\beta_i-(-1)^j\Delta_i)}{64\varepsilon_3^{-2}\Delta f^{(j)^3}}\ .
\end{aligned} \tag{A8}$$

Here $i=1,2$ and the partial derivatives of the functions $\Delta(m_1,m_2)$, $\beta(m_1,m_2)$, $\gamma(m_1,m_2)$ have the form

$$\begin{aligned}
&\Delta_i=2\beta\beta_i-4\varepsilon_3^{-1}\gamma_i\ ,\quad \Delta_{ii}=2\beta_i^2+2\beta\beta_{ii}-4\varepsilon_3^{-1}\gamma_{ii}\ ,\\
&\Delta_{12}=2\beta_1\beta_2-4\varepsilon_3^{-1}\gamma_{12}\ ,\\
&\beta_i=2(\varepsilon_i^{-1}+\varepsilon_3^{-1})m_i\ ,\quad \beta_{ii}=2(\varepsilon_i^{-1}+\varepsilon_3^{-1})\ ,\\
&\gamma_1=2m_1[2\varepsilon_1^{-1}m_1^2+(\varepsilon_1^{-1}+\varepsilon_2^{-1})m_2^2-\varepsilon_1^{-1}(\varepsilon_2^{-1}+\varepsilon_3^{-1})]\ ,\\
&\gamma_2=2m_2[2\varepsilon_2^{-1}m_2^2+(\varepsilon_1^{-1}+\varepsilon_2^{-1})m_1^2-\varepsilon_2^{-1}(\varepsilon_1^{-1}+\varepsilon_3^{-1})]\ ,\\
&\gamma_{11}=12\varepsilon_1^{-1}m_1^2+2(\varepsilon_1^{-1}+\varepsilon_2^{-1})m_2^2-2\varepsilon_1^{-1}(\varepsilon_2^{-1}+\varepsilon_3^{-1})\ ,\\
&\gamma_{22}=12\varepsilon_2^{-1}m_2^2+2(\varepsilon_1^{-1}+\varepsilon_2^{-1})m_1^2-2\varepsilon_2^{-1}(\varepsilon_1^{-1}+\varepsilon_3^{-1})\ ,\\
&\gamma_{12}=4(\varepsilon_1^{-1}+\varepsilon_2^{-1})m_1m_2\ .
\end{aligned} \tag{A9}$$

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